Notes: Dynamic Pricing in Different Valuation Models

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1 Proof of Lemma 3.11

Proof. For a small neighborhood of x^*, i.e., $(x^* - \epsilon, x^* + \epsilon)$, we have:

- Local strongly concavity of $f(x)$:

$$\exists A_1 > 0, A_1(x^* - x)^2 \leq f(x^*) - f(x).$$

- Local smoothness:

$$\exists A_2 > 0, f(x^*) - f(x) \leq A_2(x^* - x)^2.$$

For the rest part of $[0, 1]$, i.e., $\{x \in [0, 1], |x^* - x| \geq \epsilon\}$, we have $f(x^*) - f(x) > 0$. Since this is a compact set, we have: $\min f(x^*) - f(x) > 0$. Therefore, we further have:

- Let $B_1 = \min f(x^*) - f(x) > 0$, and we have:

$$B_1(x^* - x)^2 \leq f(x^*) - f(x).$$

- Let $B_2 = \frac{\max f(x^*) - f(x)}{\epsilon^2} > 0$, and we have:

$$f(x^*) - f(x) \geq B_2(x^* - x)^2.$$

Now, let $C_1 = \min\{A_1, B_1\}, C_2 = \max\{A_2, B_2\}$, and the lemma holds. □

2 Proof of Corollary 3.12

Proof. From Lemma 3.11, we have: $C_1(x^* - x)^2 \leq f(x^*) - f(x)$. Then,

- Replace x with $\frac{i}{K}$, and we get

$$f(x^*) - \mu_i \geq C_1(x^* - \frac{i}{K})^2.$$

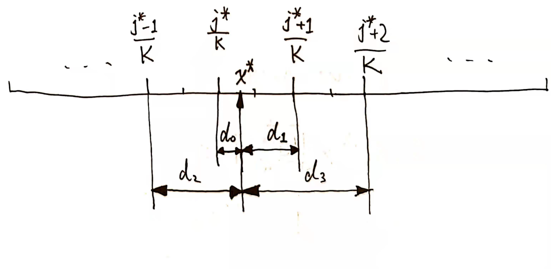

Figure 1: Corollary 3.12 Part 2

In the original paper, the authors claim a $\mu^* - \mu_i \geq C_1(x^* - \frac{i}{K})^2$, which is at least NOT a direct derivation of Lemma 3.11. In order to mend the proof, we may firstly notice that $f(x^*) - \mu^* \leq C_2 \cdot \frac{1}{K^2}$. By assuming that the smallest C_2, i.e., the exact quadratic upper bound of $f(x^*) - f(x)$, is just slightly larger than the largest C_1, i.e. the exact quadratic lower bound of $f(x^*) - f(x)$, we know that $(x^* - \frac{i}{K})^2$ is much larger than $\frac{C_2}{K^2}$ for most i, and the inequality of Corollary 3.12 holds for a new C_1.

- Denote $j* = \arg\min_j |\frac{j}{K} - x^*|$. Figure 1 illustrate the situation. We can see that $d_0 \geq 0, d_1 \geq \frac{2K}{,} d_{i+2} \geq \frac{1}{K} + d_i$. Therefore, we have $d_i \geq \frac{i}{2K}$.

□

3 Proof of Corollary 3.13

Proof. We have:

$$
\begin{aligned}
& f(x^*) - f(x) \leq C_2(x^* - x)^2 \\
\underset{x = \frac{j^*}{K}}{\overset{\uparrow}{\Rightarrow}} & f(x) - C_2(x^* - \frac{j^*}{K})^2 \leq f(\frac{j^*}{K}) \\
\underset{|x^* - \frac{j^*}{K}| \leq \frac{1}{K}}{\overset{\uparrow}{\Rightarrow}} & f(x^*) - C_2 \cdot \frac{1}{K^2} \leq \mu^*.
\end{aligned}
$$

□

4 Proof of Theorem 3.14

Proof. On the one hand, we have:

$$\begin{aligned}
\sum_{i:\mu_i<\mu^*} \frac{1}{\Delta_i} &\leq \sum_i \frac{1}{C_1 \cdot (\frac{i}{2K})^2} \\
&= \frac{4K^2}{C_1} \sum_i \frac{1}{i^2} \\
&\leq \frac{2\pi^2}{3 \cdot C_1} \sqrt{\frac{n}{\log n}}.
\end{aligned}$$

Here the first inequality comes from Corollary 3.12, and the last line comes from the fact that $\sum_{i=1\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$. On the other hand, we have:

$$\sum_{j=1}^{K} \Delta_j \leq K = (\frac{n}{\log n})^{\frac{1}{4}}.$$

Therefore, we have:

$$Reg \leq (8 \sum_{i:\mu_i<\mu^*} \frac{\log n}{\Delta_i}) + (1 + \frac{\pi^2}{3})(\sum_{j=1}^{K} \Delta_j) = O(\sqrt{n \log n}).$$

□

5 Proof of $\mathbb{E}[\rho(p^*) - \rho(x^*)]$

Proof. Recall that $p^* := \arg\max_x \rho(x)$ and that $\rho(x) = \sum_{t=1}^n x \cdot \mathbb{1}(x \leq y_t)$. Since x^* is independent to y_t's, we have: $\mathbb{E}[\rho(x^*)] = f(x^*) = \max_x f(x)$. Therefore, for any $x \leq p^*$, we have:

$$\begin{aligned}
\rho(x) &= \sum t = 1^n x \cdot \mathbb{1}(x \leq y_t) \\
&\underset{\substack{\uparrow \\ p^* \geq x}}{\geq} \sum_{t=1}^{n} x \cdot \mathbb{1}(p^* \leq y_t) \\
&= \sum_{t=1}^{n} p^* \cdot \mathbb{1}(p^* \leq y_t) - \sum_{t=1}^{n} (p^* - x) \cdot \mathbb{1}(p^* \leq y_t) \\
&\geq \rho(p^*) - n(p^* - x).
\end{aligned}$$

Hence,

$$
\begin{aligned}
\int_0^1 \Pr[\rho(x) - \rho(x^*) > \lambda] dx &\geq \int_0^1 \Pr[\rho(p^*) - n(p^* - x) - \rho(x^*) > \lambda] dx \\
&\geq \int_0^1 \Pr[\{\rho(p^*) - \rho(x^*) > 2\lambda\} \cap \{n(p^* - x) \leq \lambda\}] dx \\
&\geq \frac{\lambda}{n} \Pr[\rho(p^*) - \rho(x^*) > 2\lambda].
\end{aligned}
$$

According to Chernoff-Hoeffding Inequality (for martingales), we have:

$$
\Pr[\rho(p^*) - \rho(x^*) > \lambda] < \exp\{-\frac{\lambda^2}{2n}\}.
$$

Therefore, we have:

$$
\Pr[\rho(p^*) - \rho(x^*) > 2\lambda] \leq \min\{1, \frac{n}{\lambda} \exp(-\frac{\lambda^2}{2n})\}.
$$

Hence

$$
\begin{aligned}
\mathbb{E}[\rho(p^*) - \rho(x^*)] &\leq \int_0^{+\infty} \Pr[\rho(p^*) - \rho(x^*) > y] dy \\
&\leq \int_0^{+\infty} \min\{1, \frac{n}{y} \exp(-\frac{y^2}{2n})\} dy \\
&\leq \int_0^{\sqrt{4n \log n}} 1 dy + \int_{\sqrt{4n \log n}}^{+\infty} \frac{2n}{\sqrt{4n \log n}} \exp(-\frac{y^2}{2n}) dy \\
&= O(\sqrt{n \log n}).
\end{aligned} \tag{1}
$$

The first inequality of Equation 1 is due to the second definition of expectation: for a random variable $X \geq 0$, we have:

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^{+\infty} x P(x) dx \\
&= \int_0^{+\infty} P(x) \int_0^x 1 dy dx \\
&= \int_0^{+\infty} 1 dy \int_y^{+\infty} p(x) dx \\
&= \int_0^{+\infty} 1 dy \cdot \Pr[X \geq y] \\
&= \int_0^{+\infty} \Pr[X \geq y] dy.
\end{aligned}
$$

The last line of Equation 1 comes from the property of Gaussian distribution: $\int_t^{+\infty} \exp(-\frac{z^2}{2}) dz \leq \frac{\exp(-\frac{t^2}{2})}{t}$. Based on this observation, the second term of the second last line of Equation 1 can be upper bounded by $\frac{1}{2n \log n}$. □